



# An Interval Branch and Bound Algorithm for Parameter Estimation

Bertrand Neveu, Martin de La Gorce, Gilles Trombettoni

## ► To cite this version:

Bertrand Neveu, Martin de La Gorce, Gilles Trombettoni. An Interval Branch and Bound Algorithm for Parameter Estimation. GOW: Globla Optimization Workshop, Sep 2016, Braga, Portugal. hal-01376601

**HAL Id: hal-01376601**

**<https://hal-enpc.archives-ouvertes.fr/hal-01376601>**

Submitted on 5 Oct 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An Interval Branch and Bound Algorithm for Parameter Estimation

Bertrand Neveu,<sup>1</sup> Martin de la Gorce<sup>1</sup> and Gilles Trombettoni<sup>2</sup>

<sup>1</sup>*LIGM, Ecole des Ponts Paristech, Université Paris Est, France,* Bertrand.Neveu@enpc.fr

<sup>2</sup>*LIRMM, Université de Montpellier, France,* Gilles.Trombettoni@lirmm.fr

**Abstract** The parameter estimation problem is a widespread and challenging problem in engineering sciences consisting in computing the parameters of a parametric model that fit observed data. The computer vision community has proposed the RANSAC algorithm to deal with outliers in the observed data. This randomized algorithm is efficient but non-deterministic and therefore incomplete. Jaulin et al. propose a branch-and-contract algorithm that returns all the model instances fitting at least  $q$  observations. Assuming that at least  $q$  observed data are inliers, this algorithm achieves on the observations a relaxed intersection operator called  $q$ -intersection. First, this paper presents several improvements to Jaulin et al.'s algorithm. Second, an interval branch and bound algorithm is designed to produce a model that can explain the maximum number of observations within a given tolerance. Experiments are carried out on computer vision and image processing problems. They highlight a significant speedup w.r.t. Jaulin et al.'s interval method in 2D and 3D shape recognition problems. We have also investigated how the approach scales up in dimensions up to 7 for stereovision (estimation of essential and fundamental matrices).

**Keywords:** Interval branch and bound, Shape detection, Parameter estimation

## 1. Parameter Estimation

Parameter estimation is a difficult problem widely studied by engineering sciences. It consists in determining the  $n$  numerical parameters of a model based on  $m$  observations. Calibration or geolocation can be viewed as specific parameter estimation problems. A parameterized model is defined by an implicit equation  $f(\mathbf{x}, \mathbf{p}) = 0$ ,  $\mathbf{p} = (p_1, \dots, p_n)$  being the vector of parameters to be determined. Given a finite set of observations  $\{\mathbf{o}_1, \dots, \mathbf{o}_i, \dots, \mathbf{o}_m\}$ , we search for all the parameters vectors that are compatible with at least  $q$  of these observations. We generally have  $n \leq q \leq m$ . An observation  $\mathbf{o}_i$  is a  $d$ -dimensional vector of observed data. It is said compatible with the parameters vector  $\mathbf{p}$ , using a tolerance value  $\tau$ , when it satisfies an *observation constraint*  $-\tau \leq f(\mathbf{o}_i, \mathbf{p}) \leq +\tau$ . The *consensus set*  $C(\mathbf{p})$  is the set of observations compatible with  $\mathbf{p}$ .

$$C(\mathbf{p}) = \{\mathbf{o}_i \mid -\tau \leq f(\mathbf{o}_i, \mathbf{p}) \leq \tau\} \quad (1)$$

This parameter estimation problem becomes challenging when the function  $f$  used to define the parametric model is not linear and/or in presence of outliers. Outliers can have numerous origins, including extreme values of the noise, erroneous measurements and data reporting errors. In order to cope with outliers we search for model instances whose consensus set contains at least  $q$  elements.

This problem can be formulated as a numerical constraint satisfaction problem with  $n$  variables  $\mathbf{p} = (p_1, \dots, p_n)$  having a real interval domain, and a single constraint stating that at least  $q$  observations are compatible with the model:

$$\text{card}(C(\mathbf{p})) \geq q \quad (2)$$

The optimization version of this problem simply consists in maximizing the cardinality of the consensus set, i.e.  $q$ .

## RANSAC: parameter estimation heuristic coping with outliers

The random sample consensus algorithm (RANSAC) [1] has become a state-of-the-art tool in the computer vision and image processing communities to achieve parameter estimation robust to outliers. This stochastic algorithm proceeds by randomly sampling observations for determining a model ( $n$  observations for determining  $n$  parameters), before checking the number of other observations compatible with this model. A version of RANSAC presented in [2] is dedicated to the detection of *several* solutions (models), but it does not detect all of them. Indeed, when a solution is found, this non deterministic algorithm removes the observations involved in the consensus set before searching for a next solution.

## Deterministic interval constraint programming approach

A deterministic parameter estimation method based on interval constraint programming and robust to outliers was described in [3, 4].

We denote by  $[x_i] = [\underline{x}_i, \overline{x}_i]$  the interval/domain of a real-valued variable  $x_i$ , where  $\underline{x}_i, \overline{x}_i$  are floating-point numbers. A Cartesian product of intervals  $[\mathbf{x}] = [x_1] \times \dots \times [x_n]$  is called a (parallel-to-axes) *box*. The width of a box is given by the width  $\overline{x}_k - \underline{x}_k$  of its largest dimension  $x_k$ . Interval methods also provide *contracting operators* (called *contractors*), i.e. methods that can reduce the variable domains involved in a constraint or a set of constraints without loss of solutions. In particular, a simple forward-backward (also called HC4-revise) algorithm traverses twice the expression tree corresponding to a given constraint to contract the domains of its variables [5, 6].

The deterministic parameter estimation algorithm performs a tree search to exhaustively explore the parameter space.

- $[\mathbf{p}]$  is recursively subdivided: one variable  $p_i$  in  $\mathbf{p}$  is selected, its domain  $[p_i]$  is bisected into two sub-intervals and the two corresponding sub-boxes are explored recursively. The combinatorial process stops when a precision is reached, i.e. when the width of the current box is inferior to  $\epsilon_{sol}$ .
- At each node of the tree, a box  $[\mathbf{x}]$  is handled:
  1. A contraction is achieved using each of the  $m$  observation constraints by the forward-backward procedure, which produces an  $m$ -set  $S$  of sub-boxes of  $[\mathbf{x}]$ .
  2. The  $q$ -intersection box  $\cap^q S$  of these contracted boxes is returned.

The  $q$ -intersection operator relaxes the (generally empty) intersection of  $m$  boxes by the union of all the intersections obtained with  $q$  boxes. More formally:

**Definition 1.** *Let  $S$  be a set of boxes. The  $q$ -intersection of  $S$ , denoted by  $\cap^q S$ , is the box of smallest perimeter that encloses the set of points of  $\mathbb{R}^n$  belonging to at least  $q$  boxes.*

For instance, the box in dotted lines in Fig. 1–(b) is the 4-intersection of the  $m = 10$  two-dimensional boxes (in plain lines).

The  $q$ -intersection of boxes is a difficult problem that has been proven DP-complete in [7] and we have resorted to a non optimal projection heuristic that reasons on each dimension independently [8]. This algorithm is time  $O(nm \log(m))$ .

## 2. Improvements

We have proposed several generic improvements to the deterministic parameter estimation code, and several improvements specific to shape recognition problems.

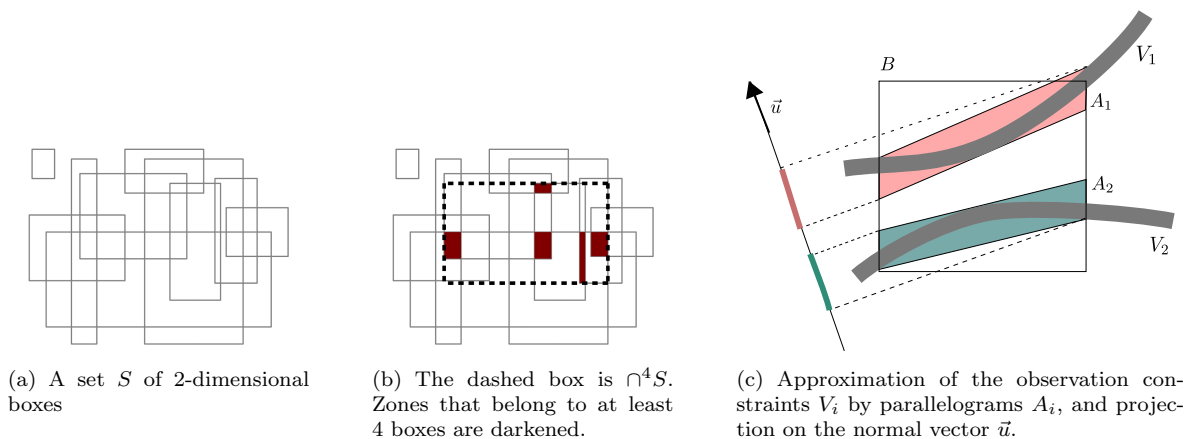


Figure 1. (b) Illustration of  $q$ -intersection for  $q = 4$ ,  $m = 2$ . (c) Q-intersection in an additional direction

## 2.1 Generic improvements

**Possible and valid observations.** In the search tree, two data structures are maintained. First, the set of *possible* observations: if an observation constraint leads to an empty box using a (forward-backward or  $q$ -intersection) contraction at a given node, this observation will be removed from the possible observations in the subtree.

Second, the number of *valid* observations is maintained by testing every possible observation (using Eq. 1) at given punctual parameters vectors inside the studied box. The valid observations form a subset of the possible observations. A stopping condition in the current branch of the search is reached when the two sets are the same.

**Q-intersection in an additional direction.** The  $q$ -intersection algorithm achieves a projection on each dimension (called  $q$ -projection) of the boxes obtained by contraction using every observation constraint.

We also perform a  $q$ -projection on an additional direction where we hope to obtain small intervals, thus favoring a failure of the  $q$ -intersection. To this end, we linearize and relax every observation constraint, and project the parallelograms obtained on the direction corresponding to the mean normal vector of the “parallelogram” gradients. See Figure 1–(c) for a 2D illustration.

In an improved version, the  $q$ -projection is achieved only in the additional direction, except in the lowest part of the search tree where all the dimensions are handled.

## 2.2 Improvements specific to applications

**Dedicated contraction.** Instead of running a general forward-backward contraction algorithm using a library for interval arithmetic computations and backward projections (e.g., implemented in *Ibex*), we can rewrite interval computations dedicated to the analytical form of observation constraints.

**Bisection strategy.** For 3D plane recognition, we bisect first the intervals of the variables corresponding to the plane normal vector and bisect the variable intervals modeling distances to the origin only when the plane normal vector intervals have reached a good precision. For 2D circle recognition, we first handle the circle center coordinates and terminate with the circle radius.

### 3. Parameter Estimation Fitting a Maximum Number of Inliers

We have also designed an interval branch and bound algorithm for parameter estimation that computes a model maximizing the consensus, i.e. maximizing the number of valid observations (inliers) of a parameterized model. Several strategies have been designed: depth-first search (and a variant) and best-first search.

Contraction using forward-backward procedures and  $q$ -intersection is performed at each iteration (node). The lower bound  $q_{min}$  of this maximization problem is given by the number of observations that have been validated in past iterations. An upper bound  $q_{max}$  of the number of inliers in a node is given by the maximum number of intersected intervals found by the  $(q_{min} + 1)$ -projection procedure in a dimension. If  $q_{max} \leq q_{min}$ , then the branch is pruned.  $q_{max}$  may be inferior to the number of possible observations in the box, in particular if the box contains several valid models.

### 4. Experiments

The algorithms are implemented in the Interval Based EXplorer (**Ibex**) [9], a free C++ library devoted to interval computing. The combination of the improvements described in Section 2 brings a significant speedup of two orders of magnitude on each tested instance of 3D plane and 2D circle detection problems and appears to be an interesting alternative to RANSAC in low dimension. These experiments suggest that our interval branch and bound algorithm can guarantee a model maximizing the number of inliers while ensuring a good performance.

### 5. Discussion

A question is whether the approach scales up in higher dimension. First experiments seem to show that the current interval branch and bound algorithm cannot cope with the fundamental matrix estimation problem (dimension 7) useful in stereovision. We will investigate whether the approach can handle parameter estimation problems of dimension 4 or 5 (essential matrix estimation [10]).

## References

- [1] M. A. Fischler and R. C. Bolles. Random Sampling Consensus: Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Commun. of the ACM*, 24(6):381–395, 1981.
- [2] R. Schnabel, R. Wahl, and R. Klein. Efficient RANSAC for Point-Cloud Shape Detection. *Computer Graphics Forum*, 26(2):214–226, 2007.
- [3] L. Jaulin and E. Walter. Guaranteed Robust Nonlinear Minimax Estimation. *IEEE Trans. on Automatic Control*, 47, 2002.
- [4] L. Jaulin and S. Bazeille. Image Shape Extraction using Interval Methods. In *Proc. of the 16th IFAC Symposium on System*, pages 378–383, 2009.
- [5] F. Benhamou, F. Goualard, L. Granvilliers, and J.-F. Puget. Revising Hull and Box Consistency. In *Proc. of International Conference on Logic Programming (ICLP)*, pages 230–244, 1999.
- [6] F. Messine. *Méthodes d’optimisation globale basées sur l’analyse d’intervalle pour la résolution des problèmes avec contraintes*. PhD thesis, LIMA-IRIT-ENSEEIH-T-INPT, Toulouse, 1997.
- [7] C. Carbonnel, G. Trombettoni, P. Vismara, and G. Chabert. Q-Intersection Algorithms for Constrained-Based Robust Parameter Estimation. In *Proc. of Conference of the Association for the Advancement of Artificial Intelligence (AAAI)*, pages 2630–2636, 2014.
- [8] P. Chew and K. Marzullo. Masking failures of Multidimensional Sensors. In *Proc. of Reliable Distributed Systems*, pages 32–41, 1991.
- [9] G. Chabert and L. Jaulin. Contractor Programming. *Artificial Intelligence*, 173:1079–1100, 2009.
- [10] J. Yang, H. Li, and Y. Jia. Optimal Essential Matrix Estimation via Inlier-Set Maximization. In *Proc. of European Conference of Computer Vision (ECCV)*, 2014.